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The L-functions of twisted Witt extensions ☆

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Abstract

The L-function of a non-degenerate twisted Witt extension is proved to be a polynomial. Its Newton polygon is proved to lie above the Hodge polygon of that extension. And the Newton polygons of the Gauss–Heilbronn sums are explicitly determined, generalizing the Stickelberger theorem.

MSC: 11L07; 14F30

Keywords: L-Functions; Exponential sums; Newton polygon

1. Introduction

Let p be a prime number, $\mathbb{Z}_p := \varprojlim \mathbb{Z}/(p^n)$ be the ring of p-adic integers, and $\mathbb{Q}_p$ its fraction field. For every positive integer n, denote by μ_n the group of nth roots of unity. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Then, for every positive power q of p, $\mathbb{F}_q := \mathbb{Z}_p[\mu_{q-1}]/(p)$ is a finite field of q elements. Every a in $\mathbb{F}_q$ has a unique representative $\omega(a)$ in $\mu_{q-1} \cup \{0\}$. It is known as the Teichmüller representative.

Let m be a positive integer, and W_m the Witt ring scheme of length m. For every ring A, the ring $W_m(A)$ is the set A^m endowed with a ring structure such that the map

$$W_m(A) \to A^m, \qquad (a_0, \dots, a_{m-1}) \mapsto (w_0, \dots, w_{m-1}), \qquad w_i = a_0^{p^i} + a_1^{p^{i-1}} p + \cdots + a_i p^i$$

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is a ring homomorphism. It is known that $W_m(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}_p[\mu_{q-1}]/(p^m)$. The isomorphism is given by

$$\iota:(a_0,\dots,a_{m-1})\mapsto\sum_{i=0}^{m-1}\omega\bigl(a_i^{p^{-i}}\bigr)p^i.$$

Fix a character Ψ_0 of $\mathbb{Z}_p/(p^m)$ of exact order p^m. Then $\Psi_q:=\Psi_0\circ\iota\circ\mathrm{Tr}_{W_m(\mathbb{F}_q)/W_m(\mathbb{F}_p)}$ is a character of $W_m(\mathbb{F}_q)$. Let f be a Witt vector of length m with coefficients in $\mathbb{F}_q[x_1^{\pm1},\dots,x_n^{\pm1}]$ and χ a character of $(\mathbb{F}_q^\times)^n$. We introduce the exponential sum

$$S(f,\chi):=(-1)^{n-1}\sum_{x\in(\mathbb{F}_q^\times)^n}\chi(x)\Psi_q\bigl(f(x)\bigr).$$

If χ is trivial, it is the exponential sum studied by Liu–Wei [LW]. If $m=1$, it is the twisted exponential sums studied by Adolphson–Sperber [AS,AS2]. In this paper, we assume that the first coordinate of f is non-zero. That obviously loses no generality.

Consider the extension

$$\mathbb{F}_q(x_1,\dots,x_n)\hookrightarrow\mathbb{F}_q\bigl(x_1^{\frac{1}{q-1}},\dots,x_n^{\frac{1}{q-1}}\bigr)(y_0,\dots,y_{m-1}),$$

where $(y_0,\dots,y_{m-1})$ satisfies the equation of Witt vectors

$$\bigl(y_0^p,\dots,y_{m-1}^p\bigr)-(y_0,\dots,y_{m-1})=f(x).$$

It is called the twisted Witt extension associated to f. Its Galois group G is isomorphic to $W_m(\mathbb{F}_p)\times(\mathbb{F}_q^\times)^n$. The isomorphism is given by

$$\kappa:g\mapsto\left(g(y_0,\dots,y_{m-1})-(y_0,\dots,y_{m-1}),\frac{g\bigl(x^{\frac{1}{q-1}}\bigr)}{x^{\frac{1}{q-1}}}\right),$$

where

$$\frac{g\bigl(x^{\frac{1}{q-1}}\bigr)}{x^{\frac{1}{q-1}}}=\left(\frac{g\bigl(x_1^{\frac{1}{q-1}}\bigr)}{x_1^{\frac{1}{q-1}}},\dots,\frac{g\bigl(x_n^{\frac{1}{q-1}}\bigr)}{x_n^{\frac{1}{q-1}}}\right).$$

Write $\mathbb{F}:=\varinjlim\mathbb{F}_{p^k}$. It is an algebraic closure of $\mathbb{F}_p$. Let $\tilde{x}$ be a class of degree k in $(\mathbb{F}^\times)^n/\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, and $\mathrm{Fr}_{\tilde{x}}$ the Frobenius element of G at $\tilde{x}$. Then we can show that

$$\kappa(\mathrm{Fr}_{\tilde{x}})=\bigl(\mathrm{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}\bigl(f(\bar{x})\bigr),\mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(\bar{x})\bigr),$$

where $\bar{x}$ is an element of $(\mathbb{F}_{q^k}^\times)^n$ representing $\tilde{x}$. Note that $\rho:=(\Psi_q\otimes\chi)\circ\kappa$ is a character of G. The Artin L-function of $\mathbb{F}_q(x_1,\dots,x_n)$ associated to ρ is

$$L_{f,\chi}(t)=\prod_{x\in(\mathbb{F}^\times)^n/\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)}\bigl(1-\rho(\mathrm{Fr}_{\tilde{x}})t^{\deg(x)}\bigr)^{(-1)^n}.$$

It is an analytic expression for the arithmetic of the twisted Witt extension associated to f. One can show that

$$L_{f,\chi}(t) = \exp\left(\sum_{k=1}^{+\infty} S_k \frac{t^k}{k}\right),$$

where $S_k := S(f, \chi \circ \mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q})$.

For every ring A, define

$$[\cdot]: A \to W_m(A), \qquad a \mapsto (a, 0, \ldots, 0),$$

and

$$V: A \to W_m(A), \qquad (a_0, \ldots, a_{m-1}) \mapsto (0, a_0, \ldots, a_{m-2}).$$

According to [LW, §0], the Witt vector f has a unique decomposition of the form

$$f = \sum_{i=0}^{m-1} \sum_{u \in I_i} V^i\big([a_{iu} x^u]\big), \quad I_i \subseteq \mathbb{Z}^n,\ a_{iu} \in \mathbb{F}_q^\times.$$

For each $i = 0, \ldots, m-1$, we denote by Δ_i the convex hull in $\mathbb{Q}^n$ of I_i and the origin. We define Δ to be the m-tuple $(\Delta_0, \ldots, \Delta_{m-1})$, and denote by Δ_∞ the convex hull in $\mathbb{Q}^n$ of $\bigcup_{i=0}^{m-1} p^{m-i-1}\Delta_i$.

In this paper, we assume that Δ_∞ generates $\mathbb{Q}^n$. Suppose that Δ_∞ generates a subspace of dimension l. In that l-dimensional subspace, choose l linearly independent integral vectors $(\alpha_{i1}, \ldots, \alpha_{in})$ $(i = 1, \ldots, l)$ that span a parallelotope of the smallest volume. Then there are integral matrices P and Q with determinants ± 1 such that $P(\alpha_{ij})_{1 \leqslant i \leqslant l, 1 \leqslant j \leqslant n} = (I, 0)Q$. It follows that we can choose integral vectors $(\alpha_{i1}, \ldots, \alpha_{in})$ $(i = l, \ldots, n)$ such that $(\alpha_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n}$ has determinant ± 1. Making the change of variables

$$\begin{cases} y_1 = x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}}, \\ \quad \vdots \\ y_l = x_n^{\alpha_{l1}} \cdots x_n^{\alpha_{ln}}, \\ z_1 = x_1^{\alpha_{l+1,1}} \cdots x_n^{\alpha_{l+1,n}}, \\ \quad \vdots \\ z_{n-l} = x_n^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}}, \end{cases}$$

we see that

$$S(f, \chi) = (-1)^{n-1} \sum_{z \in (\mathbb{F}_q^\times)^{n-l}} \chi_2(z) \sum_{y \in (\mathbb{F}_q^\times)^l} \chi_1(y) \Psi_q\big(g(y)\big)$$

for some character (χ_1, χ_2) of $(\mathbb{F}_q^\times)^n$, and some Witt vector g of the form

$$g = \sum_{i=0}^{m-1} \sum_{v \in J_i} V^i\big([b_{iv} y^v]\big), \quad J_i \subseteq \mathbb{Z}^l,\ b_{iv} \in \mathbb{F}_q^\times.$$

Therefore assuming that Δ_∞ generates $\mathbb{Q}^n$ loses no generality.

We call f non-degenerate with respect to Δ if for every face τ of Δ_∞ that does not contain 0, the system $\overline{{}_1f}^\tau = \cdots = \overline{{}_nf}^\tau = 0$ has no common solution in $(\overline{\mathbb{F}}^\times)^n$, where

$$\overline{{}_jf}^\tau = \sum_{i=0}^{m-1} \sum_{p^{m-i-1}u \in \tau} u_j a_{iu}^{p^{m-i-1}} x^{p^{m-i-1}u}.$$

In this paper we assume that f is non-degenerate. Generalizing corresponding results of Adolphson–Sperber [AS2, Corollary 2.12] and Liu–Wei [LW, Theorem 1.3], we establish the following theorem.

Theorem 1.1. *For each χ, $L_{f,\chi}$ is a polynomial.*

Since $L_{f,\chi}$ is a polynomial with coefficients in $\mathbb{Q}(\mu_{(q-1)p^m})$, it is interesting to know the prime decomposition of its reciprocal roots. Fix an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$. Knowing the prime decomposition of all reciprocal roots is equivalent to knowing the Newton polygon of $L_{f,\chi}$. The Newton polygon of a polynomial $1 + \sum_{i=0}^k \alpha_i t^i$, with respect to a valuation ord of $\overline{\mathbb{Q}}_p$, is the convex hull in $\mathbb{Q}^2$ of the points $(0,0)$ and $(i, \mathrm{ord}(\alpha_i))$ $(i = 1, \ldots, k)$. Denote by $C(\Delta_\infty)$ the cone in $\mathbb{Q}^n$ generated by Δ. There is a $\mathbb{R}_{\geqslant 0}$-linear degree function $u \mapsto \deg(u)$ on $C(\Delta_\infty)$ such that $\deg(u) = 1$ when u lies on a face of Δ_∞ that does not contain the origin. For every integer s, write $L_s(\Delta_\infty) := C(\Delta_\infty) \cap (\frac{s}{q-1} + \mathbb{Z}^n)$. There is a least positive integer D such that $D \deg L_0(\Delta_\infty) \subset \mathbb{Z}$. For every natural number k, we denote by $W_s(k)$ the number of points of degree $\frac{k}{D(q-1)}$ in $L_s(\Delta_\infty)$. Define $P_s(t) = (1 - t^{D(q-1)})^n \sum_{k=0}^{+\infty} W_s(k) t^k$. By Corollary 3.9, $P_s(t)$ is a polynomial with coefficients in the set of natural numbers. The degree-M Hodge polygon of a polynomial $\sum_{i=0}^l \alpha_i t^i$ with non-negative coefficients is the polygon in $\mathbb{Q}^2$ with vertices at the points $(0,0)$ and $(\sum_{i=0}^k \alpha_i, \sum_{i=0}^k \frac{i}{M}\alpha_i)$ $(i = 0, \ldots, l)$. From now on, we write $q = p^a$ and denote by s the integer such that, for each $x \in \mathbb{F}_q^\times$, $\chi(x) = \omega(x)^{-s}$ in $\overline{\mathbb{Q}}_p$. Generalizing corresponding results of Adolphson–Sperber [AS2, Theorem 3.17] and Liu–Wei [LW, Theorem 1.3], we establish the following theorem.

Theorem 1.2. *The Newton polygon of $L_{f,\chi}$ with respect to* ord_q *lies above the degree-$D(q-1)$ Hodge polygon of $\frac{1}{a}\sum_{i=0}^{a-1} P_{sp^i}(t)$ with the same endpoints. In particular, it is of degree $n!\,\mathrm{Vol}(\Delta_\infty)$.*

We call $S(f,\chi)$ a Gauss–Heilbronn sum if $n = 1$ and $f = \sum_{i=0}^{m-1} V^i([c_i x])$. Without loss of generality, we assume that $c_0 = 1$. If $m = 1$, the Gauss–Heilbronn sum becomes a Gauss sum. And if χ is trivial, it is a Heilbronn sum. Write $\frac{s}{q-1} = -\sum_{l=0}^{+\infty} s_l p^l$. Generalizing the Stickelberger theorem, we establish the following theorem.

Theorem 1.3. *If $n = 1$ and $f = \sum_{i=0}^{m-1} V^i([c_i x])$ with $c_0 = 1$, then the Newton polygon of $L_\chi(t)$ with respect to* ord_q *coincides with the polygon with vertices at* $(0,0)$ *and the points*

$$\left(k, \frac{(k-1)k}{2p^{m-1}} + \frac{k(s_0 + \cdots + s_{a-1})}{ap^{m-1}(p-1)}\right), \quad k = 1, \ldots, p^{m-1}.$$

Theorem 1.3 is the main contribution of this paper. In the proof of Theorem 1.3, we choose a suitable basis of the space in question so that the corresponding matrix of the Dwork operator is amenable, and then we relate that matrix to a matrix from the coefficients of the exponential function.

2. The p-adic trace formula

We prove Theorem 1.1 after establishing a p-adic trace formula relating $L_{f,\chi}(t)$ to the characteristic polynomials of an operator on p-adic spaces.

Let

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right) \in \mathbb{Z}_p[\![t]\!]$$

be the Artin–Hasse exponential series.

Lemma 2.1. [LW, Lemma 2.2] *Let l be a positive integer. Then $\pi \mapsto E(\pi)$ is a bijection from the set of roots of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ to the set of all primitive p^lth roots of unity in $\overline{\mathbb{Q}}_p$.*

By that lemma, we may choose, for each $l = 1, \ldots, m$, a unique root π_l of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ such that $E(\pi_l) = \Psi_0(1)^{p^{m-l}}$. Let π be a $D(q-1)$th root of π_m, and $O := \mathbb{Z}_p[\mu_{q-1}, \pi]$. For every $b \geqslant 0$, write

$$L_s(b) = \left\{ \sum_{u \in L_s(\Delta_\infty)} a_u x^u \colon a_u \in O,\ \mathrm{ord}_p(a_u) \geqslant b \deg(u) \right\}.$$

Define

$$E_f(x) = \prod_{i=0}^{m-1} \prod_{u \in I_i} E\big(\pi_{m-i} \omega(a_{iu}) x^u\big).$$

Lemma 2.2. [LW, Lemma 2.5] *We have $E_f \in L_0(\frac{1}{p-1})$.*

Let σ be the element of $\mathrm{Gal}(\mathbb{Q}_p[\mu_{q-1}, \pi]/\mathbb{Q}_p)$ that fixes π and becomes the p-power map when restricted to μ_{q-1}. Let σ acts on $L(b)$ coefficient-wise. For every positive integer k, write

$$E_{f,q^k}(x) = \prod_{l=0}^{ak-1} E_f^{\sigma^l}\big(x^{p^l}\big).$$

It is easy to see that

$$E_{f,q^k}(x)=\prod_{j=0}^{k-1}E_{f,q}\bigl(x^{q^j}\bigr).$$

And, by the above lemma, we have $E_{f,q^k}\in L_0(\frac{1}{p^{ak-1}(p-1)})$.

Lemma 2.3. [LW, Lemma 2.6] *For every positive integer k, we have*

$$\Psi_{q^k}\bigl(f(x)\bigr)=E_{f,q^k}\bigl(\omega(x)\bigr),\quad x\in\mathbb{F}_{q^k}.$$

Corollary 2.4. *For every positive integer k, we have*

$$S_k=(-1)^{n-1}\sum_{x\in\mu^n_{q^k-1}}x^{-d(1+q+\cdots+q^k)}E_{f,q^k}(x).$$

Write $L_s(\Delta_\infty)=C(\Delta_\infty)\cap(\frac{d}{q-1}+\mathbb{Z}^n)$. Then $L_0(\Delta_\infty)+L_s(\Delta_\infty)\subseteq L_s(\Delta_\infty)$.

Corollary 2.5. *Let k be a positive integer. If $E_{f,q^k}(x)=\sum_{u\in L_0(\Delta_\infty)}a_ux^u$, then*

$$S_k=(-1)^{n-1}\bigl(q^k-1\bigr)^n\sum_{u\in L_s(\Delta_\infty)}a_{(q^k-1)u}.$$

Define

$$B_s:=\left\{\sum_{u\in L_s(\Delta_\infty)}a_ux^u\in L_s\left(\frac{1}{p-1}\right):\ \mathrm{ord}_p(a_u)-\frac{\deg(u)}{p-1}\to+\infty\text{ if }\deg(u)\to+\infty\right\}.$$

Then B_s is a B_0-module. For every $\sum_{u\in L_s(\Delta_\infty)}a_ux^u\in B_s$, define

$$\left\|\sum_{u\in L_s(\Delta_\infty)}a_ux^u\right\|=\max_{u\in L_s(\Delta_\infty)}p^{\frac{\deg(u)}{p-1}}|a_u|_p.$$

The following lemma is easy.

Lemma 2.6. *The map $\|\cdot\|$ is a norm on B_s over O, and B_s is complete with respect to that norm.*

For every $b\geqslant 0$, write

$$L_s(b)=\left\{\sum_{u\in L_s(\Delta_\infty)}a_ux^u:\ a_u\in O,\ \mathrm{ord}_p(a_u)\geqslant b\deg(u)\right\}.$$

Then $L_s(b)$ is a $L_0(b)$-module. Define

$$\psi_p:L_s(b)\to L_{sp^{a-1}}(pb),\qquad\sum_{u\in L_s}a_ux^u\mapsto\sum_{u\in L_{sp^{a-1}}}a_{pu}x^u.$$

Lemma 2.7. *The map $\phi_p := \sigma^{-1} \circ \psi_p \circ E_f$ sends B_s to $B_{sp^{a-1}}$. We call it a Dwork operator.*

Proof. Let $g(x) \in B_s$. Then $gE_f \in L_s(\frac{1}{p-1})$. So $\psi_p(gE_f) \in L_{sp^{a-1}}(\frac{p}{p-1}) \subseteq B_{sp^{a-1}}$. The lemma is proved. □

Corollary 2.8. *The map $\phi_p^{ak} = \psi_p^{ak} \circ E_{f,q^k}$. It acts on B_s, and is O-linear.*

For every sequence $\{a_u\} \in O^{L_s(\Delta_\infty)}$, we define

$$\big\|\{a_u\}\big\| := \max_{u \in L_s(\Delta_\infty)} |a_u|_p.$$

Lemma 2.9. *The map $\|\cdot\|$ is a norm on $O^{L_s(\Delta_\infty)}$ over O, and $O^{L_s(\Delta_\infty)}$ is complete with respect to that norm.*

For each $v \in L_s(\Delta_\infty)$, define a column vector $(c_{uv})_{u \in L_{sp^{a-1}}(\Delta_\infty)}$ with coefficients in O by

$$\phi_p\big(\pi^{p^{m-1}D(q-1)\deg(v)} x^v\big) = \sum_{u \in L_{sp^{a-1}}(\Delta_\infty)} c_{uv} \pi^{p^{m-1}D(q-1)\deg(u)} x^u.$$

Lemma 2.10. *We have $\|(c_{uv})_{v \in L_s(\Delta_\infty)}\| \to 0$ if $\deg(u) \to +\infty$.*

Proof. Write

$$E_f(x) = \sum_{u \in L(\Delta_\infty)} a_u x^u, \qquad \mathrm{ord}_p(a_u) \geqslant \frac{\deg(u)}{p-1}.$$

Then

$$\phi_p\big(\pi^{p^{m-1}D(q-1)\deg(v)} x^v\big) = \sum_{u \in L_{sp^{a-1}}(\Delta_\infty)} a_{pu-v} \pi^{p^{m-1}D(q-1)(\deg(v)-\deg(u))} \pi^{p^{m-1}D(q-1)\deg(u)} x^u.$$

The lemma then follows from the fact that

$$\big\|\big\{a_{pu-v} \pi^{p^{m-1}D(q-1)(\deg(v)-\deg(u))}\big\}_{v \in L_s(\Delta_\infty)}\big\| \leqslant p^{-\deg(u)}. \qquad \square$$

Corollary 2.11. *The operator ϕ_p is completely continuous.*

By Serre [Se], for every positive integer k, the trace of ϕ_p^{ak} over O is well defined and is equal to the trace of the matrix of ϕ_p^{ak} with respect to any orthonormal basis of B_s.

Lemma 2.12. *For every positive integer k,*

$$S_k = (-1)^{n-1}\big(q^k - 1\big)^n \operatorname{Tr}_{B_s/O}\big(\phi_p^{ak}\big).$$

Proof. Let $g(x) \in B_s$. We have

$$\phi_p^{ak}(g) = \psi_p^{ak}(gE_{f,q^k}).$$

Write $E_{f,q^k}(x) = \sum_{u \in L(\Delta_\infty)} a_u x^u$. Then

$$\phi_p^{ak}\left(\pi^{p^{m-1}D(q-1)\deg(v)}x^v\right) = \sum_{u \in L_s(\Delta_\infty)} a_{q^k u - v}\pi^{p^{m-1}D(q-1)\deg(v)}x^u.$$

So the trace of ϕ_p^{ak} on B_s over O equals $\sum_{u \in L_s(\Delta_\infty)} a_{(q^k-1)u}$. The lemma then follows from Corollary 2.5.

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$. For each $l = 0, 1, \ldots, n$, write

$$K_{s,l} = \bigoplus_{1 \leqslant i_1 < \cdots < i_l \leqslant n} B_s e_{i_1} \wedge \cdots \wedge e_{i_l}$$

and define

$$\phi_{p,l} : K_{s,l} \to K_{sp^{a-1},l}, \qquad ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto p^l \phi_q(g) e_{i_1} \wedge \cdots \wedge e_{i_l}.$$

By Lemma 2.12, we have the following chain-level trace formula. □

Proposition 2.13. *For every positive integer k,*

$$S_k = \sum_{l=0}^{n} (-1)^{l+1} \operatorname{Tr}_{K_{s,l}/O}\left(\phi_{p,l}^{ak}\right).$$

Define

$$\hat{E}_f(x) := \prod_{j=0}^{+\infty} E_{f,q}\left(x^{q^j}\right).$$

And write

$$d \log \widehat{E}_f(x) = \sum_{k=1}^{n} \widehat{{}_k f} \frac{dx_k}{x_k}.$$

Lemma 2.14. [LW, Corollary 3.8] *For $k = 1, \ldots, n$, we have $\widehat{{}_k f} \in B_0$, and*

$$\widehat{{}_k f} \equiv \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} \sum_{\deg(p^{m-i-1}u)=1} u_k \omega\left(a_{iu}^{p^j}\right) \pi^{D(q-1)p^{m-1}\deg(p^j u)} x^{p^j u} \pmod{\pi B_0}.$$

By that lemma, $\hat{\partial}_j : g \mapsto (x_j \frac{\partial}{\partial x_j} + \widehat{{}_j f})g$, $j = 1, \ldots, n$, operate on B_s. Obviously, they commute with each other. So, for each $l = 1, \ldots, n$,

$$\hat{\partial} : K_{s,l} \to K_{s,l-1},$$
$$ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \hat{\partial}_{i_k}(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \quad i_1 < \cdots < i_l,$$

is well defined. We have $\hat{\partial}^2 = 0$. Thus we get a complex

$$K_{s,n} \xrightarrow{\hat{\partial}} K_{s,n-1} \xrightarrow{\hat{\partial}} \cdots \xrightarrow{\hat{\partial}} K_{s,0}.$$

It is easy to check that $\phi_{p,l-1} \circ \hat{\partial} = \hat{\partial} \circ \phi_{p,l}$. That is, $\phi_p := (\phi_{p,n}, \ldots, \phi_{p,0})$ sends the complex $(K_{s,\bullet}, \hat{\partial})$ to the complex $(K_{sp^{a-1},\bullet}, \hat{\partial})$. So ϕ_p^{ak} operates O-linearly on the complex $(K_{s,\bullet}, \hat{\partial})$. Therefore we have the following homological trace formula.

Proposition 2.15. *For every positive integer k,*

$$S_k = \sum_{l=0}^{n} (-1)^{l+1} \operatorname{Tr}_{H_l(K_{s,\bullet}, \hat{\partial})/O} \bigl(\phi_p^{ak}\bigr).$$

Equivalently,

$$L_{f,\chi}(t) = \prod_{l=0}^{n} \det{}_O \bigl(1 - \phi_p^a t \bigm| H_l(K_{s,\bullet}, \hat{\partial})\bigr)^{(-1)^l}.$$

Define $B = \bigoplus_{s=0}^{q-2} B_s$. For each $l = 0, 1, \ldots, n$, write

$$K_l = \bigoplus_{s=0}^{q-2} K_{s,l} = \bigoplus_{1 \leqslant i_1 < \cdots < i_l \leqslant n} B e_{i_1} \wedge \cdots \wedge e_{i_l}.$$

Lemma 2.16. [LW, Proposition 6.1] *The O-module $H_l(K_\bullet, \hat{\partial})$ is 0 if $l = 1, \ldots, n$, and is free of finite rank if $l = 0$.*

Form that lemma and the fact that

$$H_l(K_\bullet, \hat{\partial}) = \bigoplus_{s=0}^{q-2} H_l(K_{s,\bullet}, \hat{\partial}),$$

we deduce the following corollary.

Corollary 2.17. *The O-module $H_l(K_{s,\bullet}, \hat{\partial})$ is 0 if $l = 1, \ldots, n$, and is free of finite rank if $l = 0$.*

From that corollary and the homological trace formula, we deduce Theorem 1.1. More precisely, we have the following corollary.

Corollary 2.18. *The O-module $H_0(K_{s,\bullet}, \hat{\partial})$ is free of finite rank and*

$$L_{f,\chi}(t) = \det_O\bigl(1 - \phi_p^a t \bigm| H_0(K_{s,\bullet}, \hat{\partial})\bigr).$$

3. Bases represented by homogenous elements

We prove the following proposition.

Proposition 3.1. *The O-module $H_0(K_{s,\bullet}, \hat{\partial})$ is free of finite rank $n!\,\mathrm{Vol}(\Delta_\infty)$. Moreover, it has a basis represented a set V_s of homogenous elements such that*

$$P_s(t) = \sum_{k=0} t^k \sum_{\eta\in V_s:\ \deg(\eta)=\frac{k}{D(q-1)}} 1.$$

Define

$$\overline{B}_s := \mathbb{F}_q\bigl[x^{L_s(\Delta_\infty)}\bigr] := \Biggl\{ \sum_{u\in L_s(\Delta_\infty)} a_u x^u \colon a_u \in \mathbb{F}_q,\ a_u = 0 \text{ for all but finitely many } u \Biggr\},$$

and $\overline{B} = \bigoplus_{s=0}^{q-2} \overline{B}_s$. Then $\overline{B}_s$ is a $\overline{B}_0$-module, with the multiplication rule

$$x^u x^{u'} = \begin{cases} x^{u+u'}, & \text{if } u \text{ and } u' \text{ are cofacial,} \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$\bmod \pi : B_s \to \overline{B}_s, \qquad \sum_{u\in L_s(\Delta_\infty)} a_u \pi^{D(q-1)p^{m-1}\deg(u)} x^u \mapsto \sum_{u\in L_s(\Delta_\infty)} \bar{a}_u x^u,$$

where $\bar{a}_u = a_u + \pi O$.

Lemma 3.2. *The sequence*

$$0 \to B_s \xrightarrow{\pi} B_s \xrightarrow{\bmod \pi} \overline{B}_s \to 0$$

is exact.

For $j = 1, \ldots, n$, we define

$$\bar{\partial}_j : \overline{B}_s \to \overline{B}_s, \qquad g \mapsto \left(x_j \frac{\partial}{\partial x_j} + \overline{{}_j f} \right) g,$$

where

$$\overline{{}_j f} = \sum_{i=0}^{m-1} \sum_{k=0}^{m-i-1} \sum_{\deg(p^{m-i-1}u)=1} u_j a_{iu}^{p^k} x^{p^k u}.$$

By Lemma 2.14, we have the following lemma.

Lemma 3.3. *For $j = 1, \ldots, n$, the diagram*

$$\begin{array}{ccc} B_s & \xrightarrow{\bmod \pi} & \overline{B}_s \\ {\scriptstyle \hat{\partial}_j}\downarrow & & \downarrow{\scriptstyle \bar{\partial}_j} \\ B_s & \xrightarrow{\bmod \pi} & \overline{B}_s \end{array}$$

is commutative.

For $l = 0, \ldots, n$, we define

$$\overline{K}_{s,l} = \bigoplus_{1 \leqslant i_1 < \cdots < i_l \leqslant n} \overline{B}_s e_{i_1} \wedge \cdots \wedge e_{i_l},$$

and

$$\overline{K}_l = \bigoplus_{s=0}^{q-2} \overline{K}_{s,l} = \bigoplus_{1 \leqslant i_1 < \cdots < i_l \leqslant n} \overline{B} e_{i_1} \wedge \cdots \wedge e_{i_l}.$$

For $l = 1, \ldots, n$, we define

$$\bar{\partial} : \overline{K}_{s,l} \to \overline{K}_{s,l-1},$$

$$g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \bar{\partial}_{i_k}(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \quad i_1 < \cdots < i_l.$$

It is easy to see that the sequence

$$\overline{K}_{s,n} \xrightarrow{\bar{\partial}} \overline{K}_{s,n-1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \overline{K}_{s,0}$$

is a complex.

Proposition 3.4. *The sequence*

$$0 \to (K_{s,\bullet}, \hat{\partial}) \xrightarrow{\pi} (K_{s,\bullet}, \hat{\partial}) \xrightarrow{\bmod \pi} (\overline{K}_{s,\bullet}, \bar{\partial}) \to 0$$

is exact.

Corollary 3.5. *The $\mathbb{F}_q$-space $H_l(\overline{K}_{s,\bullet}, \bar{\partial})$ is 0 if $l = 1, \ldots, n$, and is of dimension equal to the O-rank of $H_l(K_{s,\bullet}, \hat{\partial})$ if $l = 0$.*

By that corollary, Proposition 3.1 follows from the following one.

Proposition 3.6. *The $\mathbb{F}_q$-space $H_0(\overline{K}_{s,\bullet}, \bar{\partial})$ is of dimension $n!\,\mathrm{Vol}(\Delta_\infty)$. Moreover, it has a basis represented a set $\overline{V}_s$ of homogenous elements such that*

$$P_s(t) = \sum_{k=0} t^k \sum_{\eta\in\overline{V}_s:\ \deg(\eta)=\frac{k}{D(q-1)}} 1.$$

For $j = 1, \dots, n$, we define

$$\overline{{}_jf}^0 = \sum_{i=0}^{m-1} \sum_{\deg(p^{m-i-1}u)=1} u_j a_{iu}^{p^{m-i-1}} x^{p^{m-i-1}u}.$$

For $l = 1, \dots, n$, we define

$$\begin{aligned} &\bar{\partial}^0 : \overline{K}_{s,l} \to \overline{K}_{s,l-1}, \\ &ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \overline{{}_{i_k}f}^0 ge_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \quad i_1 < \cdots < i_l. \end{aligned}$$

Then

$$\overline{K}_{s,n} \xrightarrow{\bar{\partial}^0} \overline{K}_{s,n-1} \xrightarrow{\bar{\partial}^0} \cdots \xrightarrow{\bar{\partial}^0} \overline{K}_{s,0}$$

is a complex.

Lemma 3.7. [LW, Proposition 6.6] *The $\mathbb{F}_q$-space $H_l(\overline{K}_\bullet, \bar{\partial}^0)$ is 0 if $l = 1, \dots, n$, and is of finite dimension if $l = 0$.*

Corollary 3.8. *The $\mathbb{F}_q$-space $H_l(\overline{K}_{s,\bullet}, \bar{\partial}^0)$ is 0 if $l = 1, \dots, n$, and is of finite dimension if $l = 0$.*

Corollary 3.9. *The Poincaré series of $H_l(\overline{K}_{s,\bullet}, \bar{\partial}^0)$ over $\mathbb{F}_q$ is $P_s(t)$. In particular, $P_s(t)$ is a polynomial with coefficients in the set of natural numbers.*

Modifying the argument of [Ho, 2.10, 2.13], we can prove the following lemma.

Lemma 3.10. *We have $P_s(t)|_{t=1} = n!\,\mathrm{Vol}(\Delta_\infty)$.*

Proposition 3.11. *Let V be a basis of $H_0(\overline{K}_{s,\bullet}, \bar{\partial}^0)$ over $\mathbb{F}_q$ consisting of homogeneous elements. Then V is also a basis of $H_l(\overline{K}_{s,\bullet}, \bar{\partial})$ over $\mathbb{F}_q$.*

Proof. First, we show that $\overline{K}_{s,0}$ is generated by V and $\bar{\partial}(\overline{K}_{s,1})$ over $\mathbb{F}_q$. Otherwise, among elements of $\overline{K}_{s,0}$ which are not linear combinations of elements of V and $\bar{\partial}(\overline{K}_{s,1})$, we choose one of lowest degree. We may suppose that it is of form $\bar{\partial}^0(\xi)$. Let ξ^0 be the leading term of ξ. Then $\bar{\partial}^0(\xi) - \bar{\partial}(\xi^0)$ is not a linear combination of elements of V and $\bar{\partial}(\overline{K}_{s,1})$, and is of lower degree than $\partial^0(\xi)$. This is a contradiction. Therefore $\overline{K}_{s,0}$ is generated by V and $\bar{\partial}(\overline{K}_{s,1})$ over $\mathbb{F}_q$. It remains to show that $\xi = 0$ whenever ξ belongs to $\bar{\partial}(\overline{K}_{s,1})$ and is a linear combination of elements of V. Otherwise, we may choose one element ζ of lowest degree such that $\xi = \bar{\partial}(\zeta)$. Let ζ^0 be the leading term of ζ. Then $\bar{\partial}^0(\zeta^0)$ is a linear combination of elements of V since it is the leading term of $\bar{\partial}(\zeta)$. So we have $\bar{\partial}^0(\zeta^0) = 0$. By the acyclicity of $(\overline{K}_{s,\bullet}, \bar{\partial}^0)$, $\zeta^0 = \bar{\partial}^0(\eta)$ for some η. The form $\zeta - \bar{\partial}(\eta)$ is now of lower degree than ζ, contradicting to our choice of ζ. This completes the proof of the proposition. □

4. The Newton polygon

We prove Theorem 1.2. The second statement follows from the first and Lemma 3.10. The first statement follows from the following two lemmas.

Lemma 4.1. [LW, Theorem 1.3] *The Newton polygon of $\prod_\chi L_{f,\chi}$ with respect to ord_q lies above the degree-$D(q-1)$ Hodge polygon of $\sum_{s=0}^{q-2} P_s(t)$. Moreover, their endpoints coincide.*

Lemma 4.2. *The Newton polygon of $L_{f,\chi}(t)$ with respect to ord_q lies above the degree-$D(q-1)$ Hodge polygon of $\frac{1}{a}\sum_{i=0}^{a-1} P_{sp^i}(t)$.*

That lemma follows from Corollary 2.18 and the following two lemmas.

Lemma 4.3. *The Newton polygon of $\det_O(1-\phi_p^a t; H_0(K_{s,\bullet}, \hat{\partial}))$ with respect to ord_q is obtained from the Newton polygon of $\det_{\mathbb{Z}_p[\pi]}(1-\phi_p t; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial}))$ with respect to ord_p by dividing the ordinates and abscissas by a^2.*

Lemma 4.4. *The Newton polygon of $\det_{\mathbb{Z}_p[\pi]}(1-\phi_p t; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial}))$ with respect to ord_p lies above the degree-$D(q-1)$ Hodge polygon of $a\sum_{i=0}^{a-1} P_{sp^i}(t)$.*

Proof of Lemma 4.3. Let σ acts on $O[t]$ coefficient-wise. Then

$$\det_O\bigl(1-\phi_p^a t; H_0(K_{s,\bullet}, \hat{\partial})\bigr)^\sigma = \det_O\bigl(1-\phi_p^a t; H_0(K_{sp,\bullet}, \hat{\partial})\bigr).$$

So

$$\begin{aligned}\det_{\mathbb{Z}_p[\pi]}\left(1-\phi_p^a t; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial})\right) &= \mathrm{N}_{O/\mathbb{Z}_p[\pi]} \det_O\left(1-\phi_p^a t; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial})\right) \\ &= \det_O\left(1-\phi_p^a t; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial})\right)^a.\end{aligned}$$

Hence

$$\begin{aligned}\prod_{\zeta^a=1} \det_{\mathbb{Z}_p[\pi]}\left(1-\zeta\phi_p t; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial})\right) &= \det_{\mathbb{Z}_p[\pi]}\left(1-\phi_p^a t^a; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial})\right) \\ &= \det_O\left(1-\phi_p^a t^a; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial})\right)^a.\end{aligned}$$

Note that $\det_O(1-\phi_p^a t; H_0(K_{s,\bullet},\hat\partial))$ and $\det_O(1-\phi_p^a t; H_0(K_{sp,\bullet},\hat\partial))$, being conjugate to each other, share the same Newton polygon. Lemma 4.3 then follows. □

Proof of Lemma 4.4. By Proposition 3.1, the O-module $H_0(K_{s,\bullet},\hat\partial)$ has a basis represented a set V_s of homogenous elements such that

$$P_s(t)=\sum_{k=0} t^k \sum_{\eta\in V_s:\ \deg(\eta)=\frac{k}{D(q-1)}} 1.$$

For real numbers $b>\frac{1}{p-1}$ and c, write

$$L_s(b,c)=\left\{\sum_{u\in L_s(\Delta_\infty)} a_u x^u:\ a_u\in\mathbb{Q}_p[\mu_{q-1},\pi],\ \operatorname{ord}_p(a_u)\geqslant b\deg(u)+c\right\},$$

and denote by $V_s(b,c)$ the subset of elements of $L_s(b,c)$ which are finite linear combinations of elements of V_s. The space $L_s(b,c)$ is compact with respect to the topology of coefficient-wise convergence. We claim that, if $\frac{1}{p-1}<b<\frac{p}{p-1}$, then

$$L_s(b,c)=V_s(b,c)+\sum_{k=1}^n \hat\partial_k L_s\left(b,c+b-\frac{1}{p-1}\right).$$

In fact, that claim follows from a result of Liu–Wei [LW, Proposition 8.2], which says that, if $\frac{1}{p-1}<b<\frac{p}{p-1}$, then

$$L(b,c)=V(b,c)+\sum_{k=1}^n \hat\partial_k L\left(b,c+b-\frac{1}{p-1}\right),$$

where $L(b,c)=\sum_{s=0}^{q-2} L_s(b,c)$, and $V=\sum_{s=0}^{q-2} V_s$.

Let $\lambda\in\mu_{q-1}$ such that $\lambda,\dots,\lambda^a$ is a basis of $\mathbb{Z}_p[\mu_{q-1},\pi]$ over $\mathbb{Z}_p[\pi]$. Then

$$V_\chi=\left\{\lambda^i\xi:\ i=1,\dots,a,\ \xi\in V_{sp^j},\ j=0,\dots,a-1\right\}$$

is a basis of $\bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet},\hat\partial)$ over $\mathbb{Z}_p[\pi]$. For each $\xi\in V_\chi$, we write

$$\phi_p(\xi)\equiv\sum_{\eta\in V_\chi} c_{\eta,\xi}\eta\left(\operatorname{mod}\sum_{k=1}^n \hat\partial_k B_\chi\right),\quad c_{\eta,\xi}\in\mathbb{Z}_p[\pi],$$

where $B_\chi=\sum_{i=0}^{a-1} B_{sp^i}$. Then

$$\det_{\mathbb{Z}_p[\pi]}\left(1-\phi_p t;\bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet},\hat\partial)\right)=\det\bigl(1-(c_{\eta,\xi})t\bigr).$$

By definition, $\phi_p(\xi)$ lies in the space $L(\frac{p}{p-1})$. So, by our claim, $c_{\eta,\xi}\eta$ lies in every $L(b)$ with $\frac{1}{p-1} < b < \frac{p}{p-1}$. Hence $\operatorname{ord}_p(c_{\eta,\xi}) \geqslant (b - \frac{1}{p-1})\deg(\eta)$ for every $\frac{1}{p-1} < b < \frac{p}{p-1}$. It follows that $\operatorname{ord}_p(c_{\eta,\xi}) \geqslant \deg(\eta)$. Write

$$\det\bigl(1 - (c_{\eta,\xi})t\bigr) = \sum_k \alpha_n t^n.$$

And order elements of V_χ as $\eta_1, \eta_2, \ldots$ such that $\deg(\eta_i) \leqslant \deg(\eta_{i+1})$. Then

$$\operatorname{ord}_p(\alpha_n) \geqslant \sum_{i \leqslant n} \deg(\eta_i).$$

It follows that the Newton polygon of the characteristic polynomial of $(c_{\eta,\xi})$ with respect to ord_p lies above the degree-$D(q-1)$ Hodge polygon of $a\sum_{i=0}^{a-1} P_{sp^i}(t)$. Lemma 4.4 is proved. □

5. The Gauss–Heilbronn sums

We determine the Newton polygon of $L_{f,\chi}(t)$ when $n = 1$ and $f = \sum_{i=0}^{m-1} V^i([c_i x])$ with $c_0 = 1$. That is, we prove Theorem 1.3. By Lemma 4.2, it suffices to prove the following proposition.

Proposition 5.1. *The Newton polygon of* $\det_{\mathbb{Z}_p[\pi]}(1 - \phi_p t; \bigoplus_{i=0}^{a-1} H_0(K_{sp^i,\bullet}, \hat{\partial}))$ *with respect to* ord_p *coincides with the polygon with vertices at* $(0,0)$ *and the points*

$$\left(a^2 n, \frac{a^2(n-1)n}{2p^{m-1}} + \frac{an(d_0 + \cdots + d_{a-1})}{p^{m-1}(p-1)}\right), \quad n = 1, \ldots, p^{m-1}.$$

We have $\Delta_\infty = [0, p^{m-1}]$, $\deg(p^{m-1}) = 1$, $D = p^{m-1}$, and $\overline{{}_1 f}^0 = x^{p^{m-1}}$. Write

$$\frac{s}{q-1} = -\sum_{l=0}^{+\infty} s_l p^l, \quad 0 \leqslant s_l \leqslant p-1.$$

Then, for each $l = 0, \ldots, a-1$, $\overline{B}_{sp^{a-l}} = x^{\frac{s_l + s_{l+1}p + \cdots + s_{l+a-1}p^{a-1}}{q-1}}\mathbb{F}_q[x]$,

$$\overline{V}_{sp^{a-l}} = \left\{x^u\colon u = \frac{s_l + s_{l+1}p + \cdots + s_{l+a-1}p^{a-1}}{q-1} + i,\ 0 \leqslant i \leqslant p^{m-1} - 1\right\}$$

represents a basis of $\overline{B}_{sp^{a-l}}/(\overline{{}_1 f}^0)$ over $\mathbb{F}_q$, and

$$V_{sp^{a-l}} = \left\{\pi^{(q-1)p^{m-1}u} x^u\colon u = \frac{s_l + s_{l+1}p + \cdots + s_{l+a-1}p^{a-1}}{q-1} + i,\ 0 \leqslant i \leqslant p^{m-1} - 1\right\}$$

represents a basis of $H_0(K_{sp^{a-l},\bullet}, \hat{\partial})$ over $\mathbb{Z}_p[\mu_{q-1}, \pi]$. It follows that

$$U_{sp^{a-l}} = x^{\frac{s_l + s_{l+1}p + \cdots + s_{l+a-1}p^{a-1}}{q-1}}\bigl\{(\pi_m x)^i\colon 0 \leqslant i \leqslant p^{m-1} - 1\bigr\}$$

represents a basis of $H_0(K_{sp^{a-l},\bullet},\hat{\partial})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ over $\mathbb{Q}_p[\mu_{q-1},\pi]$.

Define $A_l=(a_{ij}^{(l)}\pi_m^{(p-1)i+s_l})_{0\leqslant i,j\leqslant p^{m-1}-1}$ is the matrix of ϕ_p from $H_0(K_{sp^{a-l},\bullet},\hat{\partial})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ to $H_0(K_{sp^{a-l-1},\bullet},\hat{\partial})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ with respect to those bases. Write

$$\prod_{i=0}^{m-1}E(\pi_{m-i}c_ix)=\sum_{n=-\infty}^{+\infty}\alpha_n(\pi_m x)^n,\quad \alpha_n\in\mathbb{Z}_p[\mu_{q-1},\pi].$$

Then

$$\phi_p\bigl(x^{\frac{s_l+s_{l+1}p+\cdots+s_{l+a-1}p^{a-1}}{q-1}}(\pi_m x)^j\bigr)=x^{\frac{s_l+s_{l+1}p+\cdots+s_{l+a-1}p^{a-1}}{q-1}}\sum_{i=0}^{+\infty}\alpha_{pi-j+s_l}^{\sigma^{-1}}\pi_m^{(p-1)i+s_l}(\pi_m x)^i.$$

It follows that

$$a_{ij}^{(l)}\equiv\alpha_{pi-j+s_l}^{\sigma^{-1}}\pmod{\pi}.$$

Fix a $\lambda\in\mu_{q-1}$ such that $\lambda^1,\ldots,\lambda^a$ is a basis of $\mathbb{Z}_p[\mu_{q-1},\pi]$ over $\mathbb{Z}_p[\pi]$. Let $A_{ij}^{(l)}$ be the matrix of

$$\mathbb{Z}_p[\mu_{q-1},\pi]\to\mathbb{Z}_p[\mu_{q-1},\pi],\quad x\mapsto a_{ij}^{(l)}x$$

with respect to that basis. Then

$$\{\lambda_i\xi\colon\ i=1,\ldots,a,\ \xi\in U_{sp^{a-l}}\}$$

represents a basis of $H_0(K_{sp^{a-l},\bullet},\hat{\partial})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ over $\mathbb{Q}_p[\pi]$, and

$$A^{(l)}=\bigl(A_{ij}^{(l)}\pi_m^{(p-1)i+s_l}\bigr)_{0\leqslant i,j\leqslant p^{m-1}-1}$$

is the matrix of ϕ_p from $H_0(K_{sp^{a-l},\bullet},\hat{\partial})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ to $H_0(K_{sp^{a-l-1},\bullet},\hat{\partial})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ with respect to that basis. It follows that

$$\{\lambda_i\xi\colon\ i=1,\ldots,a,\ \xi\in U_{sp^{a-l},l=0,\ldots,a-1}\}$$

represents a basis of $\sum_{l=0}^{a-1}H_0(K_{sp^{a-l},\bullet},\hat{\partial})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ over $\mathbb{Q}_p[\pi]$, and the matrix of ϕ_p with respect to that basis is

$$\begin{pmatrix}0 & A^{(0)} & 0 & \cdots & 0\\ 0 & 0 & A^{(1)} & \cdots & 0\\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & A^{(a-2)}\\ A^{(a-1)} & 0 & 0 & \cdots & 0\end{pmatrix}.$$

So Proposition 5.1 follows from the following one.

Proposition 5.2. *The determinants of the matrices* $(A_{ij}^{(l)})_{0\leqslant i,j\leqslant n}$, $0\leqslant n\leqslant p^{m-1}-1$, *are* p*-adic units.*

Proof. The matrix $(a_{ij}^{(l)})_{0\leqslant i,j\leqslant n}$ defines an endomorphism of $\mathbb{Z}_p[\mu_{q-1},\pi]^{n+1}$ in a standard way. By the following proposition, it is an automorphism. The matrix $(A_{ij}^{(l)})_{0\leqslant i,j\leqslant n}$, being a matrix of that automorphism over $\mathbb{Z}_p[\pi]$, has determinant in $\mathbb{Z}_p[\pi]^\times$. The proposition is proved. □

Proposition 5.3. *The determinants of the matrices* $(a_{ij}^{(l)})_{0\leqslant i,j\leqslant n}$, $0\leqslant n\leqslant p^{m-1}-1$, *are* p*-adic units.*

Proof. By Lemma 5.1, it suffices to show that the determinants of the matrices $(\alpha_{pi-j+s_l})_{0\leqslant i,j\leqslant n}$ are p-adic units. Without loss of generality, we assume that $c_1=\cdots=c_{m-1}=0$. Write

$$\exp(\pi_m x)=\sum_{n=-\infty}^{+\infty} b_n(\pi_m x)^n,$$

and

$$E(\pi_m x)\exp(-\pi_m x)=\sum_{n=-\infty}^{+\infty} \beta_n(\pi_m x)^{pn}.$$

Then

$$\alpha_{pi-j+s_l}=\sum_n \beta_{i-n}b_{pn-j+s_l}.$$

So

$$(\alpha_{pi-j+s_l})_{0\leqslant i,j\leqslant n}=(\beta_{i-j})_{0\leqslant i,j\leqslant n}\times(b_{pi-j+s_l})_{0\leqslant i,j\leqslant n}.$$

Hence it suffices to show that the determinants of the matrices $(b_{pi-j+s_l})_{0\leqslant i,j\leqslant n}$ are p-adic units.

We define the matrices $(e_{ij}^{(l)})_{l\leqslant i\leqslant n,0\leqslant j\leqslant n-l}$, $l=0,\dots,n$, by setting

$$e_{ij}^{(l)}=\frac{p^{l-u}(i-u)!b_{pi-j+s_k-u}}{(i-l)!\prod_{v=u}^{l-1}(pv-j+s_k-u)},$$

where u is the largest integer $\leqslant\min\{\frac{j-s_k}{p-1}+1,l\}$. We have

$$\bigl(e_{ij}^{(0)}\bigr)_{0\leqslant i,j\leqslant n}=(b_{pi-j+s_k})_{0\leqslant i,j\leqslant n}$$

and

$$e_{ij}^{(l+1)}=\begin{cases}-e_{ij}^{(l)}+\frac{e_{lj}^{(l)}}{e_{l,j+1}^{(l)}}e_{i,j+1}^{(l)}, & \text{if } pl-j+s_k\geqslant 0,\\ e_{i,j+1}^{(l)}, & \text{otherwise.}\end{cases}$$

So, denoting by w the integer such that $p(w-1) \leqslant n-s_k < pw$, we have

$$\det\bigl(e_{ij}^{(l)}\bigr)_{l\leqslant i\leqslant n,\, 0\leqslant j\leqslant n-l} = \det\bigl(e_{ij}^{(l+1)}\bigr), \quad l = 0,\dots,w-1,$$

and

$$\det\bigl(e_{ij}^{(l)}\bigr) = \frac{p^{l-u}(l-u)!\,b_{pl-n+l+s_k-u}\det(e_{ij}^{(l+1)})}{\prod_{v=u}^{l-1}(pv-n+l+s_k-u)}, \quad l = w,\dots,n-1,$$

where u is an integer defined by $n-s_k-(p-1)u < l \leqslant n-s_k-(p-1)(u-1)$. It follows that

$$\det(b_{pi-j+s_k})_{0\leqslant i,j\leqslant n} = \det\bigl(e_{ij}^{(w)}\bigr)_{w\leqslant i\leqslant n,\, 0\leqslant j\leqslant n-w},$$

$$\det\bigl(e_{ij}^{(w)}\bigr) = \det\bigl(e_{ij}^{(n-s_k-(p-1)(w-1)+1)}\bigr)\prod_{l=w}^{n-s_k-(p-1)(w-1)}\frac{p^{l-w}(l-w)!}{(p(l-w)+p-1)!},$$

$$\det\bigl(e_{ij}^{(n-s_k-(p-1)u+1)}\bigr) = \det\bigl(e_{ij}^{(n-s_k-(p-1)(u-1)+1)}\bigr)\prod_{l=n-s_k-(p-1)u+1}^{n-s_k-(p-1)(u-1)}\frac{p^{l-u}(l-u)!}{(p(l-u)+p-1)!}, \quad 0<u<w,$$

and

$$\det\bigl(e_{ij}^{(n-s_k+1)}\bigr) = \prod_{l=n-s_k+1}^{n}\frac{p^l l!}{(pl+p-1)!}.$$

Therefore we have

$$\det(b_{pi-j+s_k})_{0\leqslant i,j\leqslant n} = \prod_{l=w}^{n}\frac{p^{l-u_l}(l-u_l)!}{(p(l-u_l)+p-1)!},$$

where u_l is defined by $n-s_k-(p-1)u_l < l \leqslant n-s_k-(p-1)(u_l-1)$. In particular, $\det(b_{pi-j+s_k})_{0\leqslant i,j\leqslant n}$ is a p-adic unit. The proposition is proved. □

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